The Competence Description in Micro 3 says:

Game Theory has become a central analytic tool in much economic theory, e.g. within industrial organization, macroeconomics, international economics, labor economics, public economics and political economics.

The course aims at giving the student knowledge of game theory, non-cooperative as well as cooperative, and its applications in economic models.

The student who successfully completed the course will learn the basic game theory and will be enabled to work further with advanced game theory. The student will also learn how economic problems, involving strategic situations, can be modeled using game theory, as well as how these models are solved. The course intention is thus, that the student through this becomes able to work with modern economic theory, for instance within the areas of within industrial organization, macroeconomics, international economics, labor economics, public economics and political economics.

In the process of the course the student will learn about

- *Static games with complete information*
- *Static games with incomplete information*
- *Dynamic games with complete information*
- *Dynamic games with incomplete information*
- *Basic cooperative game theory.*

For each of these classes of games, the student should know and understand the theory, and learn how to model and analyze some important economic issues within the respective game framework.

More specifically, the students should know the theory and be able to work with both normal and extensive form games. They should know, understand and be able to apply the concepts of dominant strategies, iterative elimination of dominant strategies, as well as mixed strategies. The students should know the central equilibrium concepts in non-cooperative game theory, such as Nash Equilibrium and further refinements: Subgame-Perfect Nash Equilibrium, Bayesian Nash Equilibrium, Perfect Bayesian Equilibrium. They should understand why these concepts are central and when they are used, and be able to apply the relevant equilibrium and solution concepts.

Furthermore, the students should acquire knowledge about a number of special games and the particular issues associated with them, such as repeated games (including infinitely repeated games), auctions and signaling games.

The students should also understand and be able to apply the solution concepts of cooperative game theory, such as the core and the Shapley value. Furthermore, the students should also learn the basics of bargaining theory.

To obtain a top mark in the course the student must be able excel in all of the areas listed above.

MICRO 3 EXAM February 2010
QUESTIONS WITH SHORT ANSWERS
(Here only the short answers are given, a good exercise should argue for these answers).

1. (a) Find *all* Nash equilibria in the following game

	L	R
T	$1,1$	$3,2$
B	$4,3$	$2,0$

Solution: The pure-strategy equilibria are (B,L), (T,R) and the mixed eq can be determined as follows:

		q	1-q
		L	R
r	T	$1,1$	$3,2$
1-r	B	$4,3$	$2,0$

Row player is indifferent between playing T and B if the column player is mixing with the weight q that satisfies

$$\begin{aligned} q+3(1-q) &= 4q+2(1-q) \Leftrightarrow \\ q &= 1/4. \end{aligned}$$

Row player's best response is

$$BR_1(q)=r^*(q)\begin{cases} =0 \text{ if } q>1/4 \text{ (strategy B)} \\ \in[0,1] \text{ if } q=1/4 \text{ (any combination of } T \text{ and } B) \\ =1 \text{ if } q<1/4 \text{ (strategy T)} \end{cases}$$

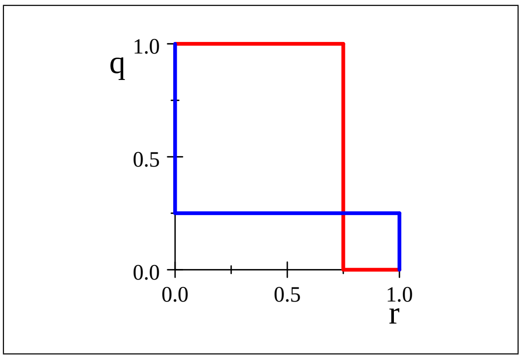
Column player is indifferent between playing L and R if the row player is mixing with the weight r that satisfies

$$\begin{aligned} r+3(1-r) &= 2r \Leftrightarrow \\ r &= 3/4. \end{aligned}$$

Column player's best response is

$$BR_2(r)=q^*(r)\begin{cases} =0 \text{ if } r>3/4 \text{ (strategy R)} \\ \in[0,1] \text{ if } r=3/4 \text{ (any combination of } L \text{ and } R) \\ =1 \text{ if } r<3/4 \text{ (strategy L)} \end{cases}$$

The intersection of BRs is (the BR of Player 1 is in blue, and the BR of player 2 is in red)



Therefore, the mixed strategy equilibrium is $((3/4, 1/4), (1/4, 3/4))$, i.e. the row player plays T with prob 3/4, and the column player plays L with prob 1/4.

(b) Solve the following game by iterated elimination of strictly dominated strategies

	t_1	t_2	t_3
s_1	$0, 2$	$1, 1$	$0, 2$
s_2	$1, 3$	$1, -1$	$4, 1$
s_3	$2, 1$	$3, 1$	$2, 0$

Solution: Elimination iteration: s_3 dominates s_1, then t_1 dominates t_3, then s_3 dominates s_2,and this is it. Solution is

	t_1	t_2
s_3	$2, 1$	$3, 1$

(c) Consider the extensive-form game represented by the game tree on Figure 1:

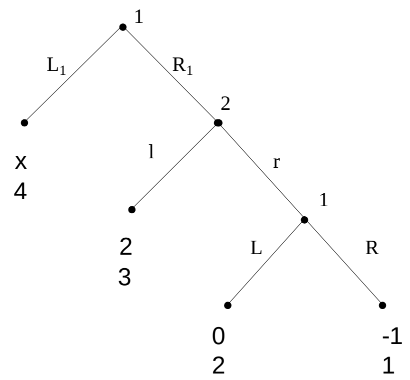


Figure 1

i. Are there any values of x under which the strategy profile (R_1L, l) is a SPNE of this game? If yes, find the respective range of x, if not, explain why.
Solution: Solve this game backwards. In the last subgame Player 1 rationally chooses L. Now consider the subgame that starts in the node controlled by Player 2. In this subgame Player 2 chooses l because he foresees the choice of Player 1 in the last subgame. If player 1 chooses R_1 in the first node and the game proceeds according to subgame-perfection, Player 1 gets 2. Therefore, in order for (R_1L, l) to be a SPNE of this game, Player 1 should get no more than 3 from choosing L_1. So, the answer is $x \leq 2$.

ii. Are there any values of x under which the strategy profile (L_1L, r) is a SPNE of this game? If yes, find the respective range of x, if not, explain why. Under which values of x is (L_1L, r) a NE of this game? Comment.
Solution: Using the same logic as above, we can see that Player 2 will never choose r in SPNE, because l brings him higher payoff. So (L_1L, r) cannot be SPNE of this game.
Let's rewrite the game in a normal form:

$$\begin{array}{ccc} & l & r \\ L_1L & x, 4 & \mathbf{x, 4} \\ L_1R & x, 4 & x, 4 \\ R_1L & 2, 3 & 0, 2 \\ R_1R & 2, 3 & -1, 1 \end{array} \tag{1}$$

In order for (L_1L, r) to be NE of this game, neither player should be interested in deviating, given the strategy of the other player. This implies the following system

$$\begin{aligned} x &\geq x \\ x &\geq 0 \\ x &\geq -1 \\ 4 &\geq 4 \end{aligned}$$

where the first 3 inequalities ensure that L_1L is the best response to r of Player 1, and the last inequality ensures that r is the best response of Player 2 to L_1L. This system implies that under $x \geq 0$ strategy profile (L_1L, r) is a NE of this game. Such a NE implies non-credible threat coming from Player 2 choosing r. As this threat is off game path, it is consistent with NE. As it is non-credible (non-rational for Player 2), it cannot be part of SPNE.

(d) Can a weakly dominated strategy be part of NE? If yes, suggest an example. If no, explain why not. (Be short and precise).

Solution: Yes it can. For example, in the normal form game (1) for $x = 0$

$$\begin{array}{ccc} & l & r \\ L_1L & 0,4 & 0,4 \\ L_1R & 0,4 & 0,4 \\ R_1L & 2,3 & 0,2 \\ R_1R & 2,3 & -1,1 \end{array} \tag{2}$$

strategy L_1L is weakly dominated by strategy R_1L. However, strategy profile (L_1L, r) is a NE of this game.

2. Consider the following game: there is a Criminal and a Sheriff. The Criminal selects the seriousness of a crime she is willing to commit, $x > 0$. The Sheriff selects the level of effort he is willing to put into catching the Criminal, $y > 0$. They make these choices simultaneously and non-cooperatively. The utility of the Criminal is given by

$$U_c = (1 - xy)x$$

where $(1 - xy)$ can be interpreted as the probability that the Criminal avoids capture, and x can be interpreted as the value of crime for the Criminal. The utility of the Sheriff is given by

$$U_s = xy - cy^2,$$

where xy represents the probability of catching the criminal, and cy^2 is the cost of effort for the Sheriff, $c > 0$.

(a) Assume the effort cost of the Sheriff, represented by parameter c, is common knowledge. What are the seriousness of crime x and Sheriff's effort level y in the Nash equilibrium of this game? In particular, what are they if $c = 1$? If $c = 4$? Explain intuitively how the equilibrium values of x and y change as c increases.

Solution: Let's find best responses of both players. The Criminal solves

$$\max_x (1 - xy)x$$

FOC is

$$1 - 2xy = 0,$$

so, the best response of the Criminal is

$$x = \frac{1}{2y}.$$

The more effort is put by the Sheriff into catching the Criminal, the less serious crime the Criminal commits.
The Sheriff solves

$$\max_y xy - cy^2$$

FOC is

$$x - 2cy = 0,$$

so the best response of the Sheriff is

$$y = \frac{x}{2c}.$$

The more serious crime is committed by the Criminal, the more effort the Sheriff is willing to put into catching him.
In equilibrium both players play best responses

$$\begin{aligned} x &= \frac{1}{2y} \\ y &= \frac{x}{2c} \end{aligned}$$

so

$$\begin{aligned} x^* &= \sqrt{c} \\ y^* &= \frac{1}{2\sqrt{c}} \end{aligned}$$

As Sheriff's effort becomes more costly, i.e. c increases, the Sheriff puts less effort into catching the criminal, and the criminal commits more serious crime. In particular, for $c = 1$

$$\begin{aligned} x^* &= 1, \\ y^* &= \frac{1}{2}. \end{aligned}$$

For $c = 4$

$$\begin{aligned} x^* &= 2, \\ y^* &= \frac{1}{4}. \end{aligned}$$

(b) Assume now that there could be two types of Sheriff: lazy, with $c_L = 4$, and hard-working, with $c_H = 1$. The Sheriff knows his own type, but the Criminal does not know which type of Sheriff she is facing. She only knows that the Sheriff can be lazy with probability 2/3 and hard-working with probability 1/3.

i. What are the seriousness of crime x, and the effort levels of the lazy/hard-working types of the Sheriff y_L, y_H in the Bayes-Nash equilibrium of this game?
Solution: In Bayes-Nash equilibrium each of the types of the Sheriff solves

$$\max_y xy - c_i y^2, \;\; i = L, H$$

so the best response of each type of the Sheriff is

$$y_i = \frac{x}{2c_i}.$$

The Criminal does not know who he faces, so he maximizes expected payoff

$$\max_x \frac{2}{3}(1 - xy_L)x + \frac{1}{3}(1 - xy_H)x.$$

FOC is

$$1 - \frac{4}{3}xy_L - \frac{1}{3}xy_H = 0.$$

The best response of the Criminal is thus

$$x = \frac{1}{\frac{4}{3}y_L + \frac{2}{3}y_H}$$

ii. So, in Bayes-Nash equilibrium each of the players (and player types) plays best response:

$$\begin{cases} y_L = \frac{x}{2c_L} = \frac{x}{8} \\ y_H = \frac{x}{2c_H} = \frac{x}{2} \\ x = \frac{1}{\frac{4}{3}y_L + \frac{2}{3}y_H} \end{cases}.$$

Solving this system we get

$$\begin{cases} x^{**} = \sqrt{2} \\ y_L^{**} = \frac{1}{4\sqrt{2}} \\ y_H^{**} = \frac{1}{\sqrt{2}} \end{cases}.$$

iii. The Sheriff knows his type both in incomplete information case of (b) and in complete information case of (a). Does the lazy type of the Sheriff exert the same effort in both cases? Why? Explain intuitively.

Solution: Both the lazy and the hard-working types of the Sheriff exert efforts in BNE which are different from their effort choice in complete information case, despite of them having full information in both cases. In particular, the lazy Sheriff exerts less effort, and the hard-working Sheriff exerts more effort than in complete information case. This is due to the fact that they account for the other party, the Criminal, not having complete information, and choosing his seriousness of crime based on expected utility. That is, the Criminal chooses the seriousness of crime which is higher than when he faces hard-working Sheriff (because the Sheriff can be lazy with some probability), and lower than when he faces lazy Sheriff (because the Sheriff can be hard-working with some probability). Both types of the Sheriff adjust their effort choices accordingly. For example, the lazy Sheriff now exerts even less effort, because he "free-rides" on the fact that the Criminal believes that with some probability the Sheriff is the hard-working type.

3. Consider the following game between Players 1 and 2

		Player 2	
		X	Y
Player 1	X	$3, 3$	$0, 4$
	Y	$4, 0$	$1, 1$

(a) Assume this game is repeated 3 times and the payoff of the resulting game is the sum of the payoffs in all three repetitions. Assume there is no time discounting. Is there a SPNE of this game, in which the payoff of either player is equal to 5? Explain your answer.

Solution: The (stage) game above has a unique NE (Y,Y). Therefore if this game is repeated a finite number of times, the SPNE of it is just a trivial repetition of stage game equilibrium. In particular, if it is repeated 3 times, the only SPNE is to play Y in each period, which yields a payoff of 3 to each player over 3 periods. Thus there is no SPNE of 3-times-repeated game in which each player gets 5.

Assume now that the game is repeated infinitely many times, and each player maximizes net present value of all future discounted payoffs. They both have the discount factor δ, where $0 < \delta < 1$.

(b) Find a range of δ such that the following strategy is supported as a subgame-perfect equilibrium of this infinitely repeated game:

- Normal phase: play X in the first period of the game or if the play in all past periods was (X, X). Otherwise revert to punishment phase forever.
- Punishment phase: Play Nash equilibrium of the stage game.

Solution: This is a standard grim trigger strategy equilibrium. Start with the normal phase. If both players stick to this strategy, each of them gets every period's payoff of 3, which results in a discounted payoff of

$$3\left(1+\delta+\delta^2+...\right)=\frac{3}{(1-\delta)}.$$

If instead, one of them, say, Player 1, chooses to deviate, then her best one-period deviation is her best response to X, that is, Y, which results in this period payoff of 4. The next period the game reverts to the Nash equilibrium forever with a payoff of 1. Therefore, its discounted payoff from deviation is

$$4+\delta\left(1+\delta+\delta^2+...\right)=4+\frac{\delta}{1-\delta}.$$

Player 1 does not deviate iff

$$\frac{3}{(1-\delta)}\geq 4+\frac{\delta}{1-\delta}.$$

Solving this inequality for δ yields

$$\delta\geq\frac{1}{3}.$$

The same is true for the second player in the normal phase.

Note that none of the players wants to deviate in the punishment phase as they play stage game NE.

So if $\delta \geq 1/3$, the strategy above is supported as a subgame-perfect equilibrium of this infinitely repeated game.

(c) Find a range of δ such that the following strategy is supported as a subgame-perfect equilibrium of this infinitely repeated game.

- Normal phase: play X in the first period of the game or if the play in the past period was (X, X) or (Y, Y). Otherwise revert to punishment phase.
- Punishment phase: Play Nash equilibrium strategy of the stage game *for 1 period*, and revert back to normal phase.

Comment on the intuition behind the difference of your answers in (b) and (c).

Solution: Start with the normal phase. If both players stick to this strategy, each of them gets every period a payoff of 3, which results in a discounted payoff of

$$3\left(1+\delta+\delta^2+...\right)=\frac{3}{(1-\delta)}.$$

If instead, one of them, say, Player 1, chooses to deviate, then her best one-period deviation is her best response to X, that is, Y, which results in this period payoff of 4. The next period the game reverts to the Nash equilibrium (Y,Y) **for one period** with a payoff of 1, and after that back to (X,X). Therefore, its discounted payoff from deviation is

$$4+\delta+3\delta^2\left(1+\delta+\delta^2+...\right)=4+\delta+\frac{3\delta^2}{1-\delta}.$$

Player 1 does not deviate iff

$$\frac{3}{(1-\delta)} \geq 4 + \delta + \frac{3\delta^2}{1-\delta}.$$

Solving this inequality for δ yields

$$\delta \geq \frac{1}{2}.$$

The same is true for the second player in the normal phase.

Now consider the punishment phase. There is no one-stage deviation that can improve the position of deviating player, as they play Nash Equilibrium in (the only actual) punishment period.

So if $\delta \geq 1/2$, the strategy above is supported as a subgame-perfect equilibrium of this infinitely repeated game.

The situations in (b) and (c) differ by the "length" of the punishment period. In (c) the punishment phase is shorter, so the incentive to deviate for each player is stronger. Therefore, in order to sustain a cooperative outcome, the players should be more patient, i.e., more interested in future payoffs, which is captured by the higher threshold for the time discounting parameter δ.

4. Three flatmates, Andreas, Bente and Carl, are planning a joint party and discuss how to divide the costs of it. Each of them has made a list of his/her guests, and there are 5 guests in Andreas' list, 7 guests in Bente's list and 5 guests in Carl's list. However, their guest lists partially overlap: there are 8 guests in the joint list of Andreas and Bente, 9 guests in the joint list of Andreas and Carl, and 8 guests in the joint list of Bente and Carl. The list of all guests to the party comprises 11 persons. They estimate the cost per guest to be DKK 100.

   (a) Think of this situation as of cooperative game and write down the values of all coalitions.

   Solution: Denote Andreas by A, Bente by B and Carl by C. Let us talk of coalition values as of costs. Then

$$\begin{aligned} v(ABC) &= 1100 \\ v(AB) &= 800 \\ v(AC) &= 900 \\ v(BC) &= 800 \\ v(A) &= 500 \\ v(B) &= 700 \\ v(C) &= 500 \end{aligned}$$

   (b) Assume that the flatmates decided to use the knowledge they have acquired in Micro 3 course and came up with an idea of sharing the costs according to Shapley value. How much will each of them pay?

   Solution: There are 6 possible orderings:

$$ABC, ACB, BAC, BCA, CAB, CBA.$$

   Let's find the marginal contributions of Andreas to there orderings. His marginal contribution to the ordering ABC is

$$m(ABC, A) = v(A) - v(\{\varnothing\}) = 500.$$

Similarly,

$$\begin{aligned}
m(ACB, A) &= v(A) - v(\{\varnothing\}) = 500, \\
m(BAC, A) &= v(BA) - v(B) = 800 - 700 = 100, \\
m(BCA, A) &= v(BCA) - v(BC) = 1100 - 800 = 300, \\
m(CAB, A) &= v(CA) - v(C) = 900 - 500 = 400 \\
m(CBA, A) &= v(CBA) - v(BC) = 1100 - 800 = 300.
\end{aligned}$$

Therefore, the Shapley Value of Andreas is

$$Sh(A) = \frac{1}{6}(500 + 500 + 100 + 300 + 400 + 300) = 2100/6 = 350$$

As Andreas and Carl are symmetric players, we can conclude that the Shapley value of Carl is also

$$Sh(C) = Sh(A) = 350.$$

Finally, as Shapley Value is efficient, the Shapley Value for Bente is

$$Sh(B) = v(ABC) - Sh(A) - Sh(C) = 1100 - 350 - 350 = 400.$$